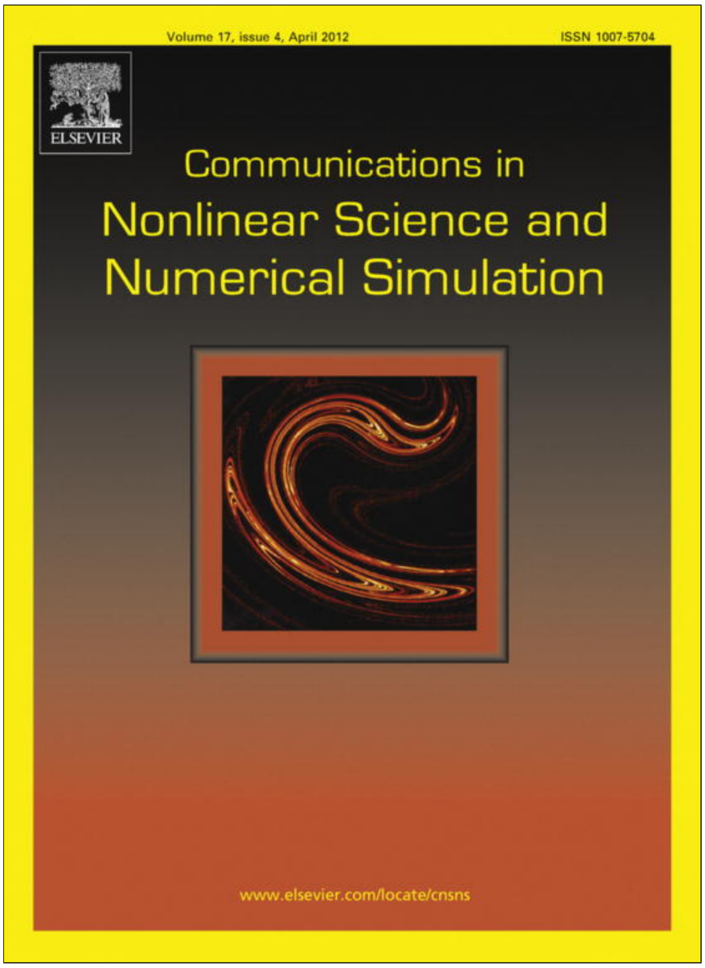
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Synchronization of coupled chaotic FitzHugh–Nagumo systems

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A B S T R A C T

This paper addresses dynamic synchronization of two FitzHugh–Nagumo (FHN) systems coupled with gap junctions. All the states of the coupled chaotic system, treating either as single-input or two-input control system, are synchronized by stabilizing their error dynamics, using simplest and locally robust control laws. The local asymptotic stability, chosen by utilizing the local Lipschitz nonlinear property of the model to address additionally the non-failure of the achieved synchronization, is ensured by formulating the matrix inequalities on the basis of Lyapunov stability theory. In the presence of disturbances, it ensures the local uniform ultimate boundedness. Furthermore, the robustness of the proposed methods is ensured against bounded disturbances besides providing the upper bound on disturbances. To the best of our knowledge, this is the computationally simplest solution for synchronization of coupled FHN modeled systems along with unique advantages of less conservative local asymptotic stability of synchronization errors with robustness. Numerical simulations are carried out to successfully validate the proposed control strategies.

1. Introduction

Chaos control and synchronization have important potential applications in several areas including biology [1], medicine [2], chemistry [3], laser technology [4], and secure communication [5] to name but a few. Not surprisingly, it has received a great deal of attention in the form of theoretical and experimental research [2–8]. For study of this phenomenon during external electrical stimulation (EES), few models have been developed, the more popular of which are discussed in detail elsewhere [9,10]. On the basis of these models, different automatic control strategies have also been reported to synchronize the coupled chaotic systems without considering the gap junctions amongst them, see [11] and references therein.

The job of synchronizing the coupled chaotic systems becomes difficult once the gap junctions are incorporated into their modular dynamics. The gap junction indeed, due to its important role in information transmittance among coupled systems, has become a research focus in synchronization and control system studies [12,13]. Researchers, thence, have paid due attention to the behavior of the gap junctions, specifically using the FitzHugh–Nagumo (FHN) model, to derive the necessary conditions, and to apply various control approaches to synchronize the coupled chaotic systems for specific applications represented by FHN model [14–16]. Generally, the proposed solutions seem to be working fine in simulations, and have even shown robustness to some extent; however, nonlinear control using feedback linearization [14–16] is computationally complex and conservative in nature. This made the synchronization achieving control response sluggish especially at the indirectly controlled state. Furthermore, these systems, in the course of feedback linearization, cancel the nonlinear terms. Hence, the implementation of such computationally inefficient and conservative control laws is complex, and nearly impossible for a network of sophisticated systems like neurons.

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Chang [17], alternatively, used a general PID approach to synchronize the states of the discrete version of a class of nonlinear delayed chaotic systems. His analysis, however, was limited to estimation of optimal PID parameters via particle swarm optimization; neither system stability nor even a proof of parameter convergence in view of stabilization was addressed. Furthermore, too many computations are required to estimate the parameters of the three PID controllers, which increases the time required for synchronization of coupled systems. Chang's proposed framework, as such, cannot be applied to more complex, sophisticated systems requiring fast synchronization.

To address these issues, we propose two simple and locally robust control strategies by treating the coupled chaotic FHN systems, considered with gap junctions coupling, as single-input and two-input control systems. The local asymptotic stability of the proposed control laws, formulated by matrix inequalities on the basis of Lyapunov stability theory, guarantees the complete synchronization of modular states. These matrix inequalities are transformable to linear matrix inequalities (LMI) for selected control parameters, which can be solved using available LMI tools. The local asymptotic stability is asserted by ensuring the states' boundedness of the addressing real systems (for example, neurons) to effectively address the non-failure of the achieved synchronization (see [18] as an example and citations therein). This introduces another advantage of utilizing a less conservative simple control strategy to resolve the design phase and implementation issues. Such a control, furthermore, allows us to incorporate other local constraints, such as robustness issues, to enhance the performance of the control system. The robustness of both strategies against disturbances is, thus, ensured by the uniform ultimate boundedness (UUB). The LMI formulation, refers to [19,20] and references therein, simplifies the selection of control parameters and automatically provides a constraint-forming matrix of quadratic Lyapunov function. This, in contrast to the conventional design-, tuning-and training-based control methodologies previously introduced [14–16,18], effectively simplifies the design stage of a synchronization strategy.

It is worth mentioning that the single-input scheme responds faster and is more robust to the direct controlling state besides confirming the asymptotic stability and robust performance to both of the states. Conversely, as obvious, the two-input control approach provides fast synchronization and identical robust performance to both the controlling states but with a necessity of individual states as controllable. Thence, the effective utilization of the proposed strategies depends on the controllable states of the specific application to which the model is representing. Furthermore, a tradeoff amongst the schemes is also countable even for the applications providing individual controllable states.

Besides the minor advantages, the main contributions of the paper are summarized underneath.

(i) Synchronization control for local asymptotic stabilization of the error dynamics of Lipschitzian nonlinear FHN systems coupled with gap junctions.
(ii) The local stability is chosen to address additionally the non-failure of the achieved synchronization by ensuring the states' boundedness.
(iii) The robust performance is ensured against bounded disturbances, upper bound to which is also provided in relation to the control parameters, by the UUB.
(iv) Simplified selection of control parameters and constraint-forming matrix of quadratic Lyapunov function, contrary to the previously reported design-, tuning-, and training-based control methodologies.
(v) To the best of our knowledge, first time addressing the simplest single-input and simplest two-input control approaches for complete synchronization of coupled FHN systems with ensured states' boundedness and an extra feature of robustness against bounded disturbances.

The paper is organized as follows. Section 2 states the synchronization problem and translates it into the problem of stabilization of the error dynamics. Section 3 formulates two control strategies to guarantee the asymptotic stability of the error dynamics with robust assurance. Section 4 verifies the proposed control strategies by means of comparative simulation results. Section 5 assesses the findings on the basis of benchmarked initial conditions. Section 6 provides concluding remarks.

2. Problem formulation

Considerable efforts both theoretical and experimental have been devoted to the study of the gap-junctions response to EES, following the pioneering work of Hodgkin and Huxley [21] of finding the cable model. The cable model is nonlinear, and its general form interestingly exhibits a variety of behaviors such as complex chaotic attractors and resonances, to simpler stationary states, periodic orbits, solitary waves, and limit cycles. Furthermore, it behaves complexly in the presence of periodic forcing. The FHN model of FitzHugh [22] and Nagumo et al. [23], one of the simplified forms of the Hodgkin and Huxley theory, therefore was chosen to prove the proposed synchronization theory for complex chaotic systems in the presence of periodic forcing. The schematic diagram in Fig. 1 illustrates a sophisticated application of FHN model; the crux scenario of weakly coupled neurons requiring deep brain stimulation (DBS) (i.e., a form of EES) to synchronize and to enhance the signal strength for the desired movement initiation or continuation task [24]. The general form of coupled master–slave FHN modular systems, as depicted by Yanagita et al. [25], is

$$\begin{cases} \dot{x}_{11} = x_{11}(x_{11}-1)(1-rx_{11}) - x_{21} - g(x_{11}-x_{12}) + (a/\omega)\cos\omega t, \\ \dot{x}_{21} = bx_{11} - \upsilon x_{21}, \end{cases} \tag{1a}$$

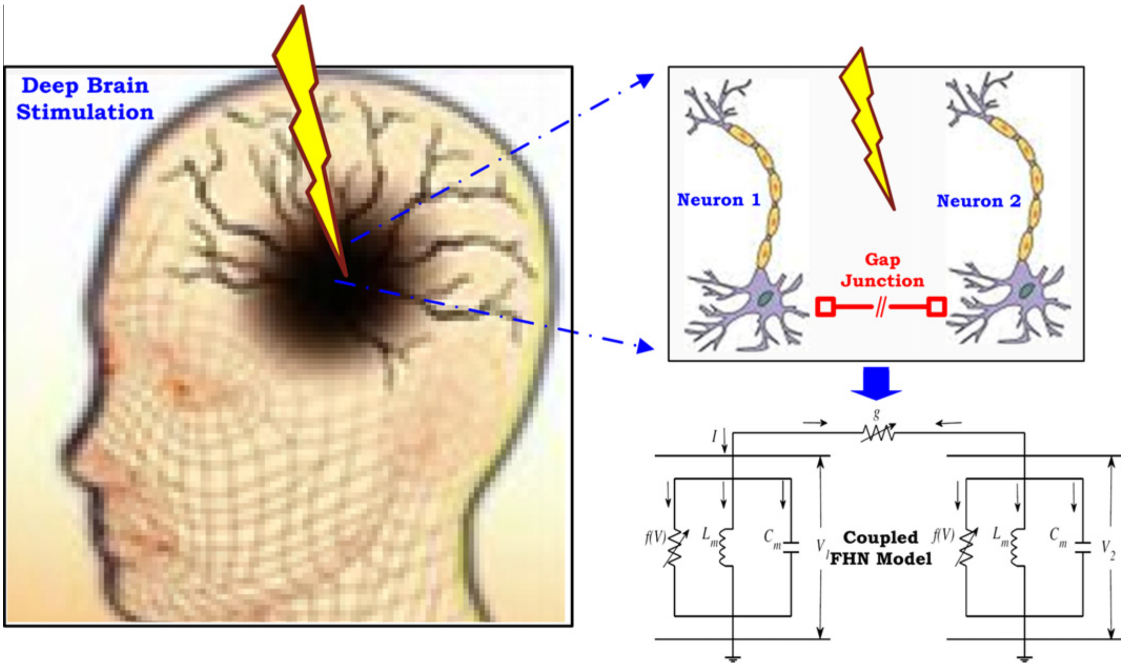

Fig. 1. DBS effect on weakly coupled neurons, modeled as FHN systems.

$$\begin{cases} \dot{x}_{12} = x_{12}(x_{12}-1)(1-rx_{12}) - x_{22} - g(x_{12}-x_{11}) + (a/\omega)\cos\omega t, \\ \dot{x}_{22} = bx_{12} - vx_{22}, \end{cases} \tag{1b}$$

where x_{11} and x_{21} are the normalized states of the master-, x_{12} and x_{22} are the normalized states of the slave-FHN systems, respectively, parameter g represents the strength of gap junctions between the master and slave systems, and $(a/\omega)\cos\omega t$ represents the external stimulation current with angular frequency ω at time t. The amplitude a and angular frequency ω are taken to be dimensionless [14,15]. The parameters of the coupled model (1) are set to

$$r = 10, \quad v = 0.1, \quad g = 0.01, \quad \omega = 0.28\pi, \quad b = 1, \quad \text{and } a = 0.1, \tag{2}$$

with the initial conditions as

$$x_{11}(0) = -0.1, \quad x_{21}(0) = -0.1, \quad x_{12}(0) = 0.3, \quad x_{22}(0) = 0.3. \tag{3}$$

The rationale for choosing such a typical set of initial conditions is to evaluate the synchronizing speed and robust performance tradeoff, amongst the proposed methodologies, and in contrast to the pervious findings. Detail validations will be carried out in Section 5, where we will discuss the pros and cons of our proposed strategies for synchronizing the coupled system (1). Fig. 2 shows the behavior of two coupled chaotic FHN systems, illustrating non-synchronous states and corresponding non-zero states' errors. It is worth noting that all of the states are bounded showing the physical limits of modeled practical systems, like activation and recovery voltages of neurons [12]. The effective error bound of the nonlinear states can be formulated as

$$\|x_{11} - x_{12}\|\beta \leqslant 1, \tag{4}$$

where $\beta \in R^{+}$, and $\|\cdot\|$ denotes the Euclidian norm. Let the state errors of coupled system (1) be defined as

$$\begin{cases} e_1 = (x_{11} - x_{12}), \\ e_2 = (x_{21} - x_{22}). \end{cases} \tag{5}$$

The derivative of (5) along (1) yields the following error dynamics.

$$\begin{cases} \dot{e}_1 = -(1+2g)e_1 - e_2 + (1+r)(x_{11}^2 - x_{12}^2) - r(x_{11}^3 - x_{12}^3), \\ \dot{e}_2 = be_1 - ve_2. \end{cases} \tag{6}$$

It is worth mentioning that the nonlinearity $f(x) = (1+r)(x_{11}^2 - x_{12}^2) - r(x_{11}^3 - x_{12}^3)$ is locally Lipschitz [26,27] for all $x_{11}, x_{12} \in R$ satisfying (4), with a Lipschitz constant $L \geqslant 0$, such that

$$\left\|\frac{\partial f}{\partial x_{11}}(x_{11})\right\|, \quad \left\|\frac{\partial f}{\partial x_{12}}(x_{12})\right\| \leqslant \|L\|, \tag{7}$$

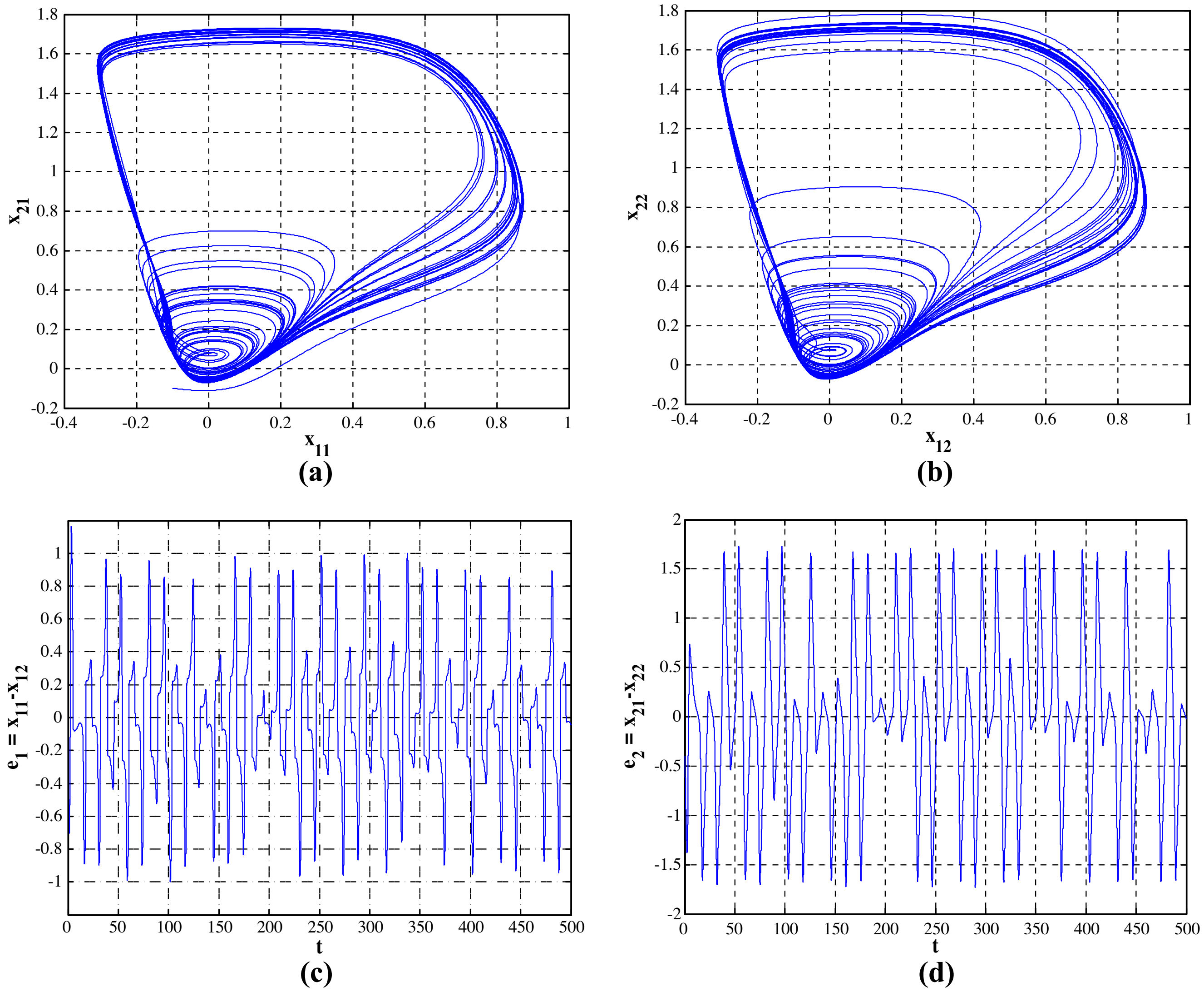

Fig. 2. Dynamics of coupled chaotic FHN model (1). (a), State space portrait of chaotic master FHN system. (b), State space portrait of chaotic slave FHN system. (c), Synchronization error, $x_{11} - x_{12}$. (d), Synchronization error, $x_{21} - x_{22}$.

satisfying

$$\left\| (1+r)\left(x_{11}^2 - x_{12}^2\right) - r\left(x_{11}^3 - x_{12}^3\right) \right\| \leqslant \|Le_1\|. \tag{8}$$

Hence, the synchronization problem of the coupled chaotic system (1) has been transformed to the problem of stabilization at the origin. Now, the goal is to appropriately actuate the coupled system (1) by applying external control input(s) such that $\lim_{t\to\infty} \|e\| = 0$ holds [28,29]. Furthermore, the bounded states of the coupled system, as described in (4), facilitate the design of a simplest and efficient controller with robust performance.

3. Simplest control methodologies with robustness

3.1. Single-input control approach

The single-input control of the coupled chaotic FHN modular system(1) addresses the chaos synchronization of practical systems having only one controllable state, x_{12} here. The single-input control form of coupled system (1) with disturbance-sensitive states will be

$$\begin{cases} \dot{x}_{11} = x_{11}(x_{11}-1)(1-rx_{11}) - x_{21} - g(x_{11}-x_{12}) + (a/\omega)\cos\omega t + d_1, \\ \dot{x}_{21} = bx_{11} - \upsilon x_{21}, \end{cases} \tag{9a}$$

$$\begin{cases} \dot{x}_{12} = x_{12}(x_{12}-1)(1-rx_{12}) - x_{22} - g(x_{12}-x_{11}) + (a/\omega)\cos\omega t + d_3 + u_1, \\ \dot{x}_{22} = bx_{12} - \upsilon x_{22}, \end{cases} \tag{9b}$$

where the terms d_1 and d_3 represent the disturbance signals at the controllable states of the master and slave systems (i.e., x_{11} and x_{12}), respectively.

The proposed simplest single-input control law is given by

$$\begin{cases} u_1 = C_1 e_1 + e_3, \\ \dot{e}_3 = C_2 e_1, \end{cases} \tag{10}$$

where C_1 and C_2 are the controller parameters to be determined. The error dynamics (6) for single-input stabilizing system (9) by introducing the proposed single control input u_1 of (10) become

$$\begin{cases} \dot{e}_1 = -(1+2g+C_1)e_1 - e_2 - e_3 + (1+r)\left(x_{11}^2 - x_{12}^2\right) - r\left(x_{11}^3 - x_{12}^3\right) + (d_1 - d_3), \\ \dot{e}_2 = be_1 - ve_2, \\ \dot{e}_3 = C_2 e_1. \end{cases} \tag{11}$$

Rewriting (11) in matrix form gives

$$\dot{e} = Ae + \varphi + d, \tag{12}$$

where

$$e = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}, \quad \varphi = \begin{bmatrix} (1+r)\left(x_{11}^2 - x_{12}^2\right) - r\left(x_{11}^3 - x_{12}^3\right) \\ 0 \\ 0 \end{bmatrix}, \quad d = \begin{bmatrix} d_1 - d_3 \\ 0 \\ 0 \end{bmatrix},$$

and

$$A = \begin{bmatrix} -(1+2g+C_1) & -1 & -1 \\ b & -v & 0 \\ C_2 & 0 & 0 \end{bmatrix}.$$

Note that φ is locally Lipschitz as defined in (8) and satisfies the following inequality.

$$\varphi^T \varphi \leqslant e^T F^T F e, \tag{13}$$

where

$$F = \begin{bmatrix} L & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Assumption 1. There exists a positive constant $d_{\max}$ such that $d^T d \leqslant d_{\max}^2$.

Theorem 1. *Given that system (9) satisfies Assumption 1, if there exist proper values of C_1 and C_2 such that the matrix inequalities*

$$P > 0, \quad p_{11} \leqslant \beta, \text{ and } \begin{bmatrix} A^T P + PA + F^T F & P \\ * & -I \end{bmatrix} = -S < 0 \tag{14}$$

are solvable with initial conditions satisfying $(x_{11}(0) - x_{12}(0))^2 p_{11} \leqslant 1$, where p_{11} represents the first row and first column entry of $P = P^T > 0$. Then the control law given by Eq. (10) *ensures the following:*

(i) *The local asymptotic stability with states always satisfying the bound $\|x_{11} - x_{12}\| p_{11} \leqslant 1$ enclosed by $\|x_{11} - x_{12}\|\beta \leqslant 1$, if $d = 0$, and*
(ii) *the locally uniformly ultimately bounded stability with states always satisfying $\|x_{11} - x_{12}\|\beta \leqslant 1$, if $d \neq 0$ and*
(iii) *the robustness against bounded disturbances ($d_{max} \leqslant \lambda_{min}/(2p_{max}\beta)$) by maximizing λ_{min}.*

Proof. Define the following Lyapunov function candidate

$$V = e^T P e. \tag{15}$$

The derivative of V along error system (12) implies

$$\dot{V} = e^T (A^T P + PA) e + e^T P \varphi + \varphi^T P e + e^T P d + d^T P e. \tag{16}$$

From (13) and (16), we have

$$\dot{V} \leqslant e^T (A^T P + PA + F^T F) e + e^T P \varphi + \varphi^T P e - \varphi^T \varphi + e^T P d + d^T P e. \tag{17}$$

Rewriting (17) in matrix form yields

$$\dot{V} \leqslant -\psi^T S\psi + e^T P d + d^T P e, \quad (18)$$

where $\psi = \begin{bmatrix} e^T & \varphi^T \end{bmatrix}^T$ and S is the same as in (14). Here, two cases arise:

Case I ($d = 0$): This makes (18) to

$$\dot{V} \leqslant -\psi^T S\psi. \quad (19)$$

Therefore, $\dot{V}(t) < 0$ implies two things.

(a) The inequality $(x_{11} - x_{12})^2 p_{11} < (x_{11}(0) - x_{12}(0))^2 p_{11}$ is satisfied. Additionally, the inequality $p_{11} \geqslant \beta$ ensures that $\|x_{11} - x_{12}\| p_{11} \leqslant 1$ is a subset of $\|x_{11} - x_{12}\| \beta \leqslant 1$. Hence, the synchronization error (0,0,0) is asymptotically stable in the region $\|x_{11} - x_{12}\| p_{11} \leqslant 1$ enclosed by $\|x_{11} - x_{12}\| \beta \leqslant 1$ for $t \geqslant 0$. Note that p_{11} can be minimized to get the maximum stability region.
(b) The inequality $S < 0$ ensures the asymptotic convergence of error e to zero.

Hence, the local error trajectory (12) asymptotically converges to zero. This completes the proof of statement (i).

Case II ($d \neq 0$): Appropriate simplifications transform (18) to

$$\dot{V} \leqslant -2\lambda_{\min}\|\psi\|^2 + 4p_{\max} d_{\max}\|\psi\| = -\|\psi\|[2\lambda_{\min}\|\psi\| - 4p_{\max} d_{\max}], \quad (20)$$

where $\lambda_{\min}$is the minimum eigenvalue of $S/2$, $p_{\max} = \max\{p_{11}, p_{12}, \ldots, p_{1n}\}$, and $\{p_{11}, p_{12}, \ldots, p_{1n}\}$ are the elements of the first row of matrix P of order n, which is $n = 3$ in this particular case. Two further cases arise here:

Case II (a). The right side of (20) is negative as long as the term inside the square bracket is positive, which implies

$$\|\psi\| > \frac{2p_{\max} d_{\max}}{\lambda_{\min}}. \quad (21)$$

Thus, $\dot{V} < 0$ is valid outside the compact set. Hence, the synchronization error is asymptotically stable for the region $\| x_{11} - x_{12}\| p_{11} \leqslant 1$ enclosed by $\|x_{11} - x_{12}\| \beta \leqslant 1$ for $t \geqslant 0$, as was proved in Case I (a).

Case II (b). Contrary situation to Case II (a); we have

$$\|\psi\|^2 = e^T e + \phi^T \phi \leqslant \frac{4p_{\max}^2 d_{\max}^2}{\lambda_{\min}^2}, \quad (22)$$

which leads to attain the error bounded condition

$$e^T e \leqslant \frac{4p_{\max}^2 d_{\max}^2}{\lambda_{\min}^2}, \quad (23)$$

and trivially to

$$e_1^2 \leqslant \frac{4p_{\max}^2 d_{\max}^2}{\lambda_{\min}^2}. \quad (24)$$

Further reduced form to this, when multiplied with β, yields the nonlinear states' error bounded condition $\|x_{11} - x_{12}\| \beta \leqslant (2p_{\max} d_{\max}\beta)/\lambda_{\min}$. Thus, to always ensure the effective error boundedness of the nonlinear states, $\|x_{11} - x_{12}\| \beta \leqslant 1$, the condition $(2p_{\max}\, d_{\max}\beta)/\lambda_{\min} \leqslant 1$ has been imposed which reveals the relation of the upper bound of the controlled disturbances with the control parameters as

$$d_{\max} \leqslant \frac{\lambda_{\min}}{2p_{\max}\beta}. \quad (25)$$

Hence, $\dot{V} < 0$ is valid outside a compact set. This reveals the robustness in terms of UUB on $\|\psi\|$ (i.e. e_1 and e_2) in consensus with the standard Lyapunov theorem extension [30,31] utilized by various researchers (see [15] as an example); and effectively ensures the boundedness of the states in $\|x_{11} - x_{12}\| \beta \leqslant 1$, if $d_{\max} \leqslant \lambda_{\min}/(2p_{\max}\beta)$. This completes the proof of statement (ii).

The proposed strategy provides the robustness against disturbances in two ways.

a. By maximizing the minimum eigenvalue of $S/2$, $\lambda_{\min}$, we can increase the maximum allowable limit, the upper bound $d_{\max}$, of the controlled disturbances in consent to (25).
b. For a fixed value of $d_{\max}$, maximization of $\lambda_{\min}$ decreases the error bound in accordance to (23) and (24).

This completes the proof of statement (iii) and effectively proves Theorem 1 in its entirety. □

3.2. Two-input control approach

This is the general case of coupled chaotic FHN modular system (1), specifically addressing the chaos synchronization of corresponding practical systems having both states directly controllable. The two-input control form of coupled system (1) with disturbance-sensitive states will be

$$\begin{cases} \dot{x}_{11} = x_{11}(x_{11}-1)(1-rx_{11}) - x_{21} - g(x_{11}-x_{12}) + (a/\omega)\cos\omega t + d_1, \\ \dot{x}_{21} = bx_{11} - \upsilon x_{21} + d_2, \end{cases} \tag{26a}$$

$$\begin{cases} \dot{x}_{12} = x_{12}(x_{12}-1)(1-rx_{12}) - x_{22} - g(x_{12}-x_{11}) + (a/\omega)\cos\omega t + d_3 + u_1, \\ \dot{x}_{22} = bx_{12} - \upsilon x_{22} + d_4 + u_2, \end{cases} \tag{26b}$$

where the terms d_1 and d_3 represent the disturbance signals at the x_{11} and x_{12} states, respectively (as was addressed in previous sub-section), and similarly the terms d_2 and d_4 represent the disturbance signals at the x_{21} and x_{22} states of the master and slave systems, respectively.

The proposed simplest two-input control law is given by

$$\begin{cases} u_1 = C_1 e_1 + e_3, \\ \dot{e}_3 = C_2 e_1, \end{cases} \tag{27a}$$

$$\begin{cases} u_2 = C_3 e_2 + e_4, \\ \dot{e}_4 = C_4 e_2, \end{cases} \tag{27b}$$

where C_j ($j = 1,\ldots,4$) are the controller's parameters that need to be determined appropriately to obtain the desired performance.

Remark 1. It is noteworthy that the design of our unified control laws (10) and (27) is much simpler than that developed in [14–16]. Such control laws, with their superior computational efficiency, are advantageous to real-time synchronization of multiple chaotic coupled FHN modular systems.

Writing error dynamics (6) for two-input stabilizing system (26) by introducing the proposed multiple control inputs u_1 and u_2 from (27) yields

$$\begin{cases} \dot{e}_1 = -(1+2g+C_1)e_1 - e_2 - e_3 + (1+r)(x_{11}^2 - x_{12}^2) - r(x_{11}^3 - x_{12}^3) + (d_1 - d_3), \\ \dot{e}_2 = be_1 - (\upsilon + C_3)e_2 - e_4 + (d_2 - d_4), \\ \dot{e}_3 = C_2 e_1, \\ \dot{e}_4 = C_4 e_2. \end{cases} \tag{28}$$

Rewriting in matrix form as

$$\dot{e} = Ae + \varphi + d, \tag{29}$$

where

$$e = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix}, \quad \varphi = \begin{bmatrix} (r+1)(x_{11}^2 - x_{12}^2) - r(x_{11}^3 - x_{12}^3) \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad d = \begin{bmatrix} d_1 - d_3 \\ d_2 - d_4 \\ 0 \\ 0 \end{bmatrix},$$

and

$$A = \begin{bmatrix} -(1+2g)+C_1 & -1 & -1 & 0 \\ b & -(\upsilon + C_3) & 0 & -1 \\ C_2 & 0 & 0 & 0 \\ 0 & C_4 & 0 & 0 \end{bmatrix}$$

with φ being locally Lipschitz as defined in (8), having form

$$\varphi^T\varphi \leqslant e^T F^T F e, \tag{30}$$

where

$$F = \begin{bmatrix} L & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Theorem 2. *Given that system (26) satisfies Assumption 1, if there exist proper values of C_j (j = 1,...,4), such that the following matrix inequalities*

$$P > 0, \quad p_{11} \leqslant \beta, \text{ and } \begin{bmatrix} A^TP + PA + F^TF & P \\ * & -I \end{bmatrix} = -S < 0, \tag{31}$$

are solvable with initial conditions satisfying $(x_{11}(0) - x_{12}(0))^2 p_{11} \leqslant 1$, where p_{11} represents first row and first column entry of $P = P^T > 0$. Then the control law given by equation (27) ensures the following:

(i) *The local asymptotic stability with states always satisfying the bound $\|x_{11} - x_{12}\| p_{11} \leqslant 1$ enclosed by $\|x_{11} - x_{12}\|\beta \leqslant 1$, if $d = 0$, and*
(ii) *the locally uniformly ultimately bounded stability with states always satisfying $\|x_{11} - x_{12}\|\beta \leqslant 1$, if $d \neq 0$ and*
(iii) *the robustness against bounded disturbances ($d_{max} \leqslant \lambda_{min}/(2p_{max}\beta)$) by maximizing λ_{min}.*

Proof. Same as that of Theorem 1. □

Remark 2. The proposed methodology ensures the boundedness of the states for synchronization of coupled locally Lipchitz FHN chaotic systems even in the presence of disturbances. It also provides the upper bound on disturbances for which locally uniformly ultimately bounded stability is ensured with states satisfying $\|x_{11} - x_{12}\|\beta \leqslant 1, \forall t \geqslant 0$. Due to this feature, the suitable control parameters ensuring the boundedness of the states in addition to disturbance rejection can be selected for the synchronization of the coupled FHN systems.

Remark 3. The local instead of global asymptotic stability, selected by ensuring the boundedness of the states, makes control laws (10) and (27) less conservative than previously reported synchronization methodologies [14,15]. This effectively allows for incorporation of the robustness against bounded disturbances, if any, and, consequently, enhances control system performance within the region.

Remark 4. It is interesting to find that the proposed matrix inequality based control methodology, solvable by LMI-tools for known control parameters C, abridges the design further by providing simplified parameter selection criteria. The automatic finding of constraint-forming matrix of the quadratic Lyapunov function reduces the efforts required for classical controller design methodologies reported by [14–16].

4. Simulation results

In this section, numerical simulations are performed to synthesize the proposed strategies. The same set of model parameters as provided in (2), and the typically selected initial conditions (3), are used throughout this section.

4.1. Single-input control

The proposed single-input control law (10) with parameters $C_1 = 200$, and $C_2 = 50$ viable the LMI constraints of Theorem 1 as strictly feasible by providing the following positive definite matrix P, calculated by MATLAB LMI Toolbox solver, after 11 iterations.

$$P = \begin{bmatrix} 2.3877 & 0.0408 & 0.0542 \\ 0.0408 & 0.2491 & 0.1463 \\ 0.0542 & 0.1463 & 0.2016 \end{bmatrix}. \tag{32}$$

To illustrate the synchronization of two coupled chaotic systems without the channel noise, the disturbance sources of single-input stabilization model (9) are set to zero; that is, $d_1, d_3 = 0$. Fig. 3 shows the single-input-controlled responses of (9) while the controller is activating at normalized time 100, in order to illustrate the transient behavior. It is apparent from Fig. 3: the two coupled systems exhibit their own chaotic dynamic behaviors, and are not synchronized until the application of the control signal. Referring to the magnified transient response of the stabilizing error in Fig. 3(b) and (d), the states' errors rapidly converge to zero, once the control signal is bring to function. Consequently, the slave system begins to behave synchronously in relation to the master one, and thus demonstrating the identical behavior of the coupled system.

The robustness of the proposed single-input control is evaluated by adding the high frequency disturbances $d_1 = 0.1 \cos(3.5t)$ and $d_3 = 0.1 \sin(3.5t)$ at the stabilizing states x_{11} and x_{12}, respectively, at the normalized time 300 in the presence of previously activated (at time 100) control signal u_1. Fig. 4, the chronological version of Fig. 3, illustrates the efficiency of the single-input control in limiting the effect of added disturbances d_1 and d_3. Appropriate segments of Fig. 4(b) and (d) are magnified to highlight the resulting small values of UUB on the state errors. As illustrated by Fig. 4, this strategy effectively

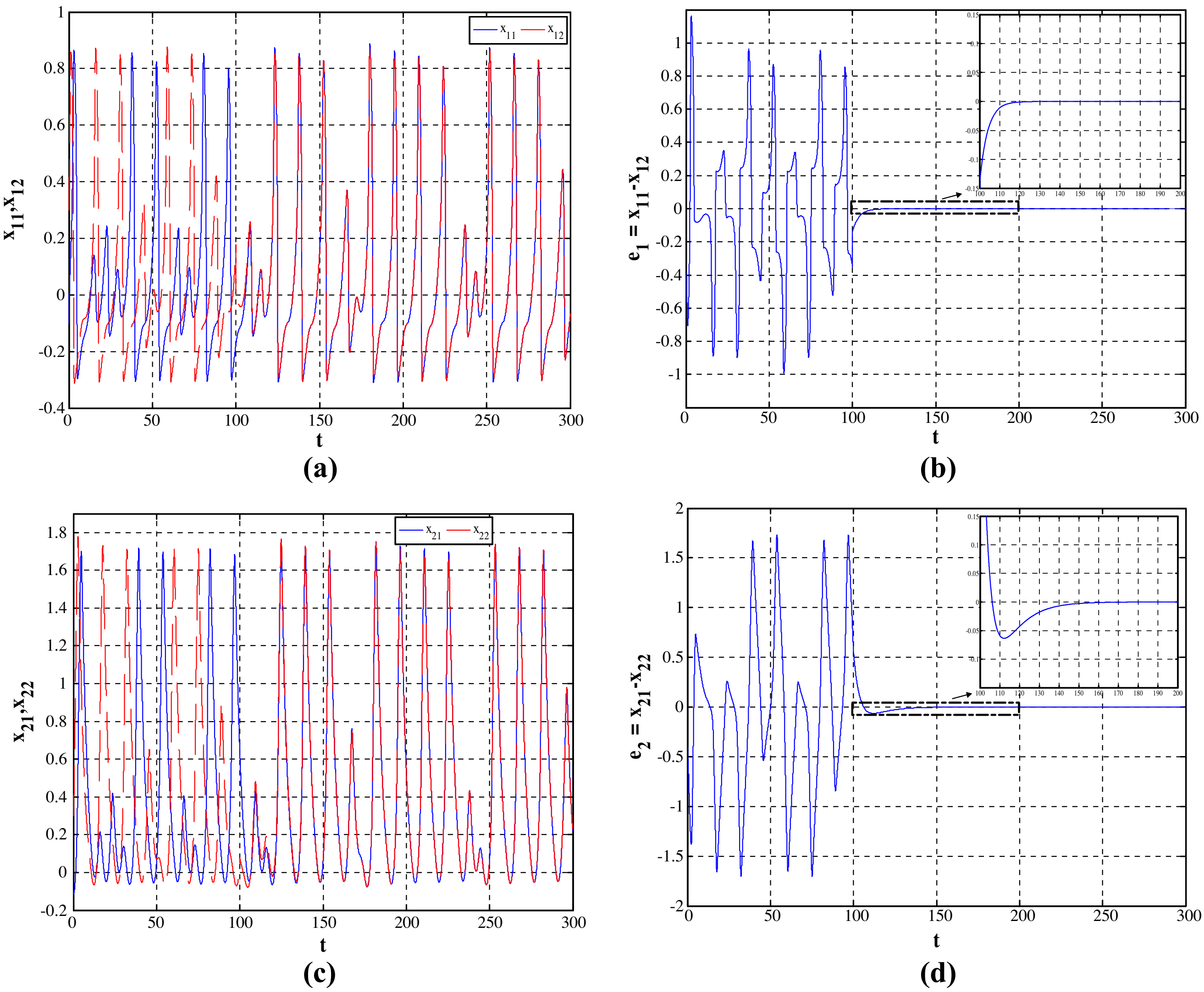

Fig. 3. Single-input controller based synchronizing response of coupled chaotic system with $d_1 = d_3 = 0$. (a), Responses of first states (x_{11},x_{12}). (b), Synchronization error e_1. (c), Responses of second states (x_{21},x_{22}). (d), Synchronization error e_2.

assures that the disturbances have almost no deleterious effect on the stable synchronization of the coupled chaotic system, verifying the identical behavior of the single-input controlled coupled system even in the presence of disturbances.

4.2. Two-input control

Before analyzing the two-input control strategy, let us investigate the single-input control mechanism further in the presence of channel noise (disturbances) coming from all of the states of the system (9), which is common for some specific applications. For this purpose, disturbances $d_2 = 0.1\cos(3.5t)$ and $d_4 = 0.1\sin(3.5t)$ are also added to the coupled system (9), at the indirectly stabilizing states x_{21} and x_{22}, respectively, beside being retaining the previously added disturbances $d_1 = 0.1\cos(3.5t)$ and $d_3 = 0.1\sin(3.5t)$ at x_{11} and x_{12} states, respectively. It is observed that the synchronization remains intact in the stabilizing states, refer to Fig. 4(a) and Fig. 5(a). The single-input control provides better robust performance against the disturbances of directly stabilizing states as compare to the disturbances at the indirectly stabilizing states, owing to its not achieving a small enough UUB on e_2, as compared with what was achieved on e_1, see Fig. 4(b) and Fig. 5(b). This lack of robustness bound on the indirectly stabilizing state motivated us to utilize the two-input stabilizer. This is absolutely practical because the system allows us, most of the time, to actuate both states simultaneously to get better performance in the presence of disturbances at both the states.

Let us now analyze two-input control strategy (26) for robust synchronization of all of the states of the coupled chaotic system by means of an efficiently small UUB on both synchronization errors in the presence of all of the sources of disturbances. The parameters of proposed two-input controller (27) satisfying the strict feasibility of the LMI constraints of Theorem 2, are selected as $C_1 = 200$, $C_2 = 50$, $C_3 = 25$, and $C_4 = 20$. With these, the solution of Theorem 2 yields the following positive definite matrix P, calculated by MATLAB LMI Toolbox solver, after 4 iterations.

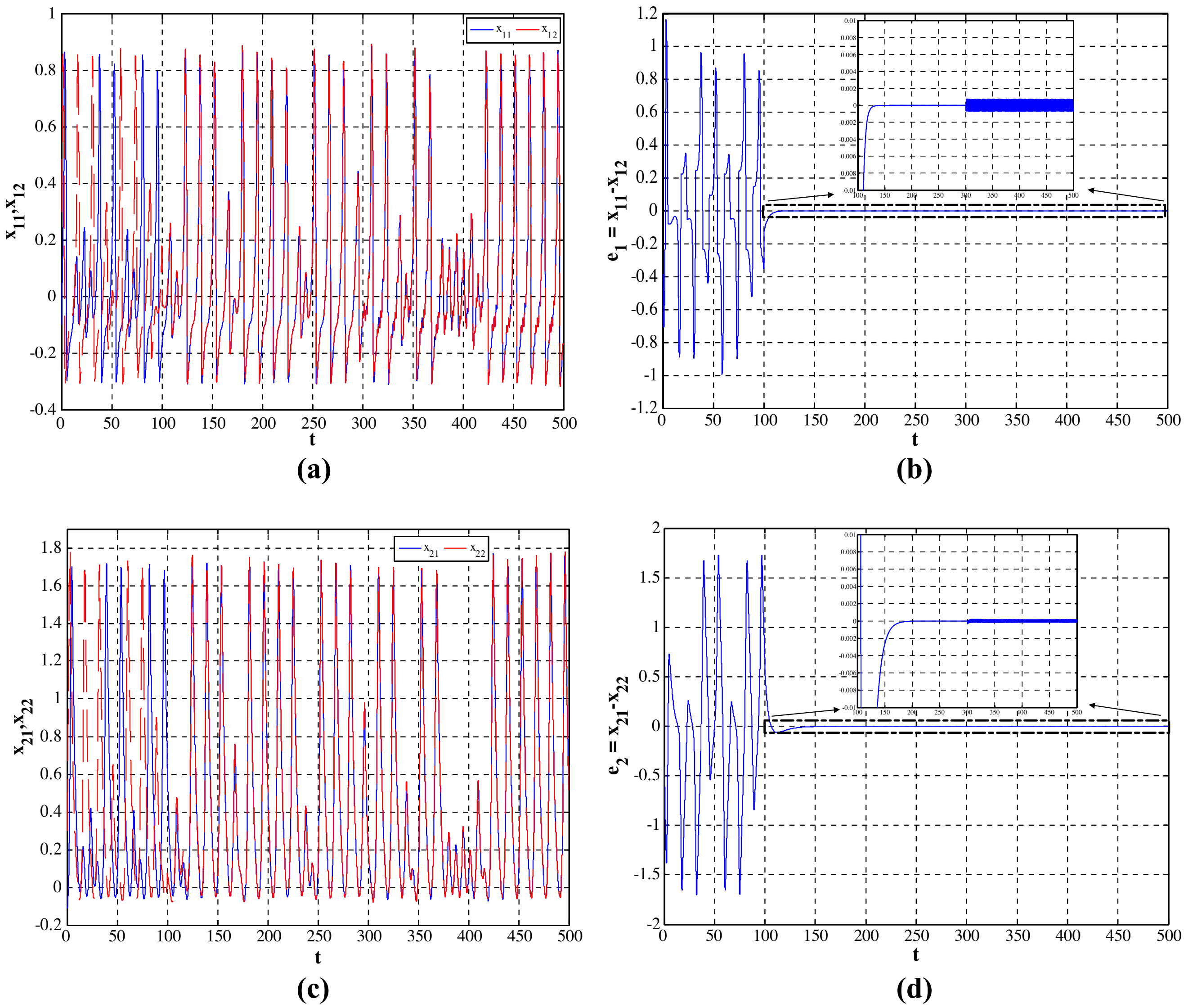

Fig. 4. Single-input controller based synchronizing response of coupled chaotic system with d_1 and d_3. (a), Responses of first states (x_{11},x_{12}). (b), Synchronization error e_1. (c), Responses of second states (x_{21},x_{22}). (d), Synchronization error e_2.

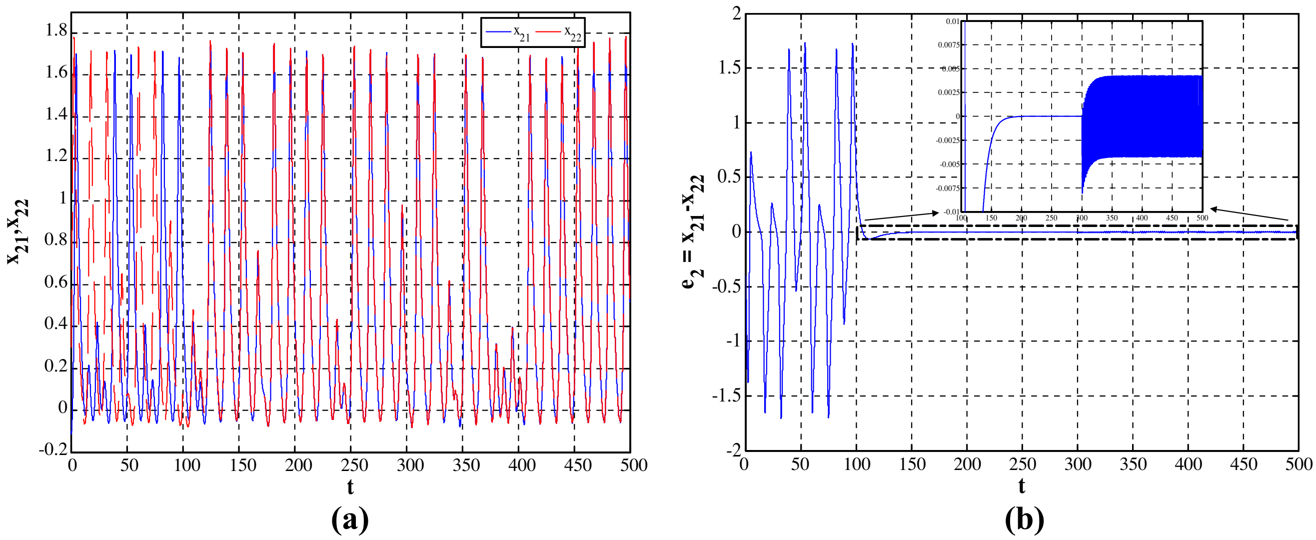

Fig. 5. Single-input controller based synchronizing response of uncontrolled state of coupled chaotic system with d_1 to d_4. (a), Responses of second states (x_{21},x_{22}). (b), Synchronization error e_2.

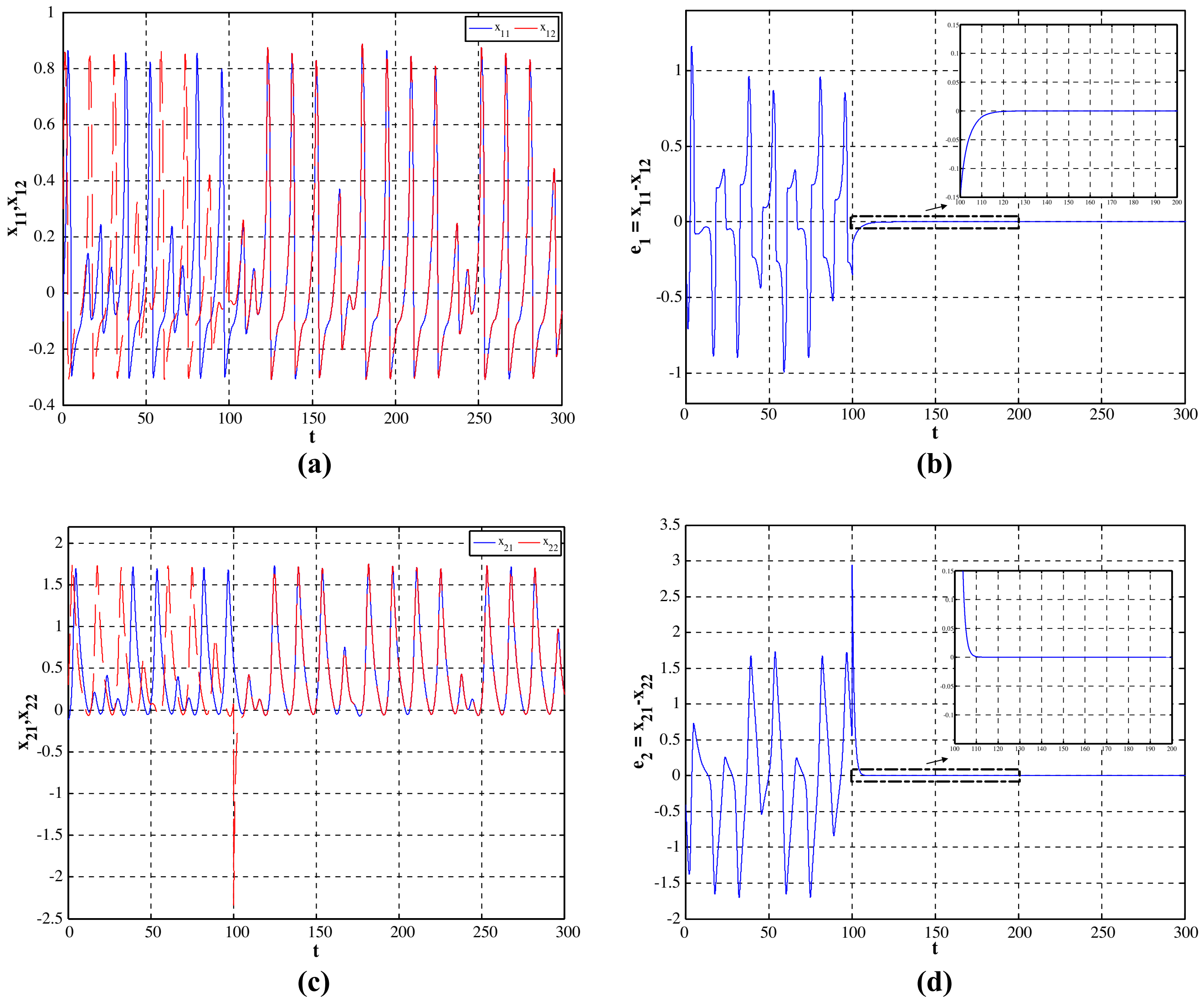

Fig. 6. Two-input controller based synchronizing response of coupled chaotic system with zero disturbances (d_1,d_2,d_3, and d_4). (a), Responses of first states (x_{11},x_{12}). (b), Synchronization error e_1. (c), Responses of second states (x_{21},x_{22}). (d), Synchronization error e_2.

$$P = \begin{bmatrix} 0.8861 & -0.0011 & 0.0731 & 0.0030 \\ -0.0011 & 0.6123 & -0.0003 & 0.4529 \\ 0.0731 & -0.0003 & 0.3083 & 0.0030 \\ 0.0030 & 0.4529 & 0.0030 & 0.5755 \end{bmatrix}. \tag{33}$$

The channel noise, specifically all of the disturbance sources, are set to zero initially; that is, $d_1,\ldots,d_4 = 0$. Analogously to Fig. 3, Fig. 6 demonstrates the stabilization of state errors and, consequently, synchronization, after the transient observing delayed activation of the proposed two-input control signals. It is also worth noting that this approach speeds up the synchronization of x_{22} with x_{21} as well, unlike the case with the single-input control strategy, plotted in Fig. 3.

To evaluate the robustness of the proposed two-input controller, the high-frequency disturbances $d_1 = 0.1\ \cos(3.5t)$, $d_2 = 0.01\ \cos(3.5t)$, $d_3 = 0.1\ \sin(3.5t)$, and $d_4 = 0.01\ \sin(3.5t)$ are introduced at the normalized simulation time 300. Fig. 7, the chronological version of Fig. 6, demonstrates how the proposed two-input controller limits the effect of added disturbances d_1 to d_4. The relevant areas of Fig. 7(b) and (d) are magnified to highlight the resulting small values of UUB on the state errors achieved by maximizing the $\lambda_{\min}$ in accordance to (23) and (24). As demonstrated by Fig. 7, this strategy assures that disturbances have almost no effect on the stability of the achieved synchronization, verifying the identical behavior of the two-input-controlled coupled system, even in the presence of all of the sources of disturbance.

5. Discussion

This paper presented noncomplex and computationally efficient techniques to synchronize two coupled chaotic FHN systems, called master and slave systems. The adopted strategy stabilizes the error dynamics of the two states of the coupled

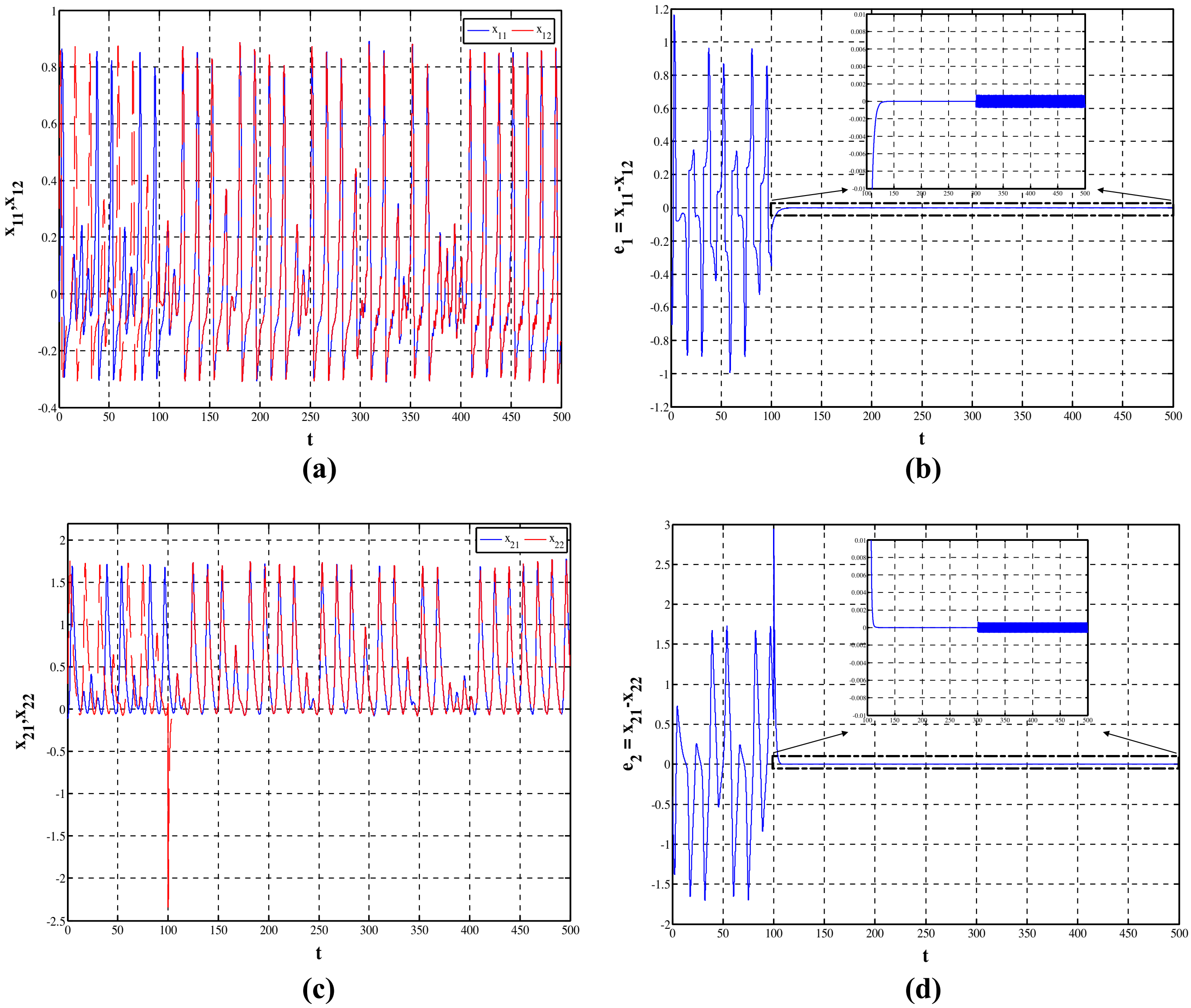

Fig. 7. Two-input controller based synchronizing response of coupled chaotic system with nonzero disturbances (d_1, d_2, d_3, and d_4). (a), Responses of first states (x_{11}, x_{12}). (b), Synchronization error e_1. (**c**), Responses of second states (x_{21}, x_{22}). (d), Synchronization error e_2.

model, effectively synchronizing them, through two simplified control approaches, namely single-input and two-input. The efficiencies of the two approaches are illustrated in Figs. 3–7, detailed descriptions of which have already been provided in Simulation Results (Section 4).

It should be noted that the effect of the simplest single-input control strategy is relatively slow at the un-activated state of the coupled system, somehow similar, but with simplest approach, to the results reported in almost all of the previously published papers having conservative and complex control approaches (see [14,15], as well as citations therein). It was not observed there, because the selected initial conditions were already situated at the synchronization line of the state space portrait on the $x_{21} - x_{22}$ plane. Thus, measuring the reliability (speed and precision) of the two proposed strategies, against each other and against all of the previously reported techniques, was the main reason behind the selection of similar model parameters but unique initial conditions in (2) and (3) respectively. Consequently, the single-input scheme responds faster and is more robust to the direct controlling state besides confirming the asymptotic stability and robust performance to both the states. This problem of slow and less robust behavior with regard to the indirectly activated state is completely resolved by the two-input control strategy, which actuates both states simultaneously, to obtain fast synchronization of both the states with identical robust performance.

6. Conclusion

In this paper, synchronization of two coupled FHN modular systems for effective synchronization of most of the physical chaotic systems of various fields (chemistry, medicine, biology, laser technology, and secure communication) was considered. The robust synchronization was achieved by stabilizing the error dynamics of the corresponding states of the coupled locally Lipchitz FHN chaotic systems; local asymptotic stability in the absence of disturbances, and locally uniformly ultimately bounded stability in the presence of bounded disturbances, upper bound to which was also formulated in relation to the control parameters. Two approaches were proposed: The single-input scheme responds faster and is more robust to the direct controlling state besides confirming the asymptotic stability and robust performance to both the states, The two-input control approach provides fast synchronization and identical robust performance to both the controlling states but with a necessity of individual states as controllable. Thence, the effective utilization of the proposed strategies depends on the controllable states of the specific application to which the model is representing. Furthermore, a tradeoff amongst the schemes is also countable even for the applications providing individual controllable states. Both stratagems make the practical implementation of coupled chaotic FHN modular systems a straightforward and easy task, owing to their advantageous simplicity (there are no nonlinear or gap junction terms), computational efficiency (they have a simple mathematical form utilizing rare floating parameters and derivative terms), and the paradigm meditation (they provides robust performance against the disturbances, allowable upper bound to which are directly related to the control parameters). Simulation results demonstrated the proposed strategies' effectiveness.

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